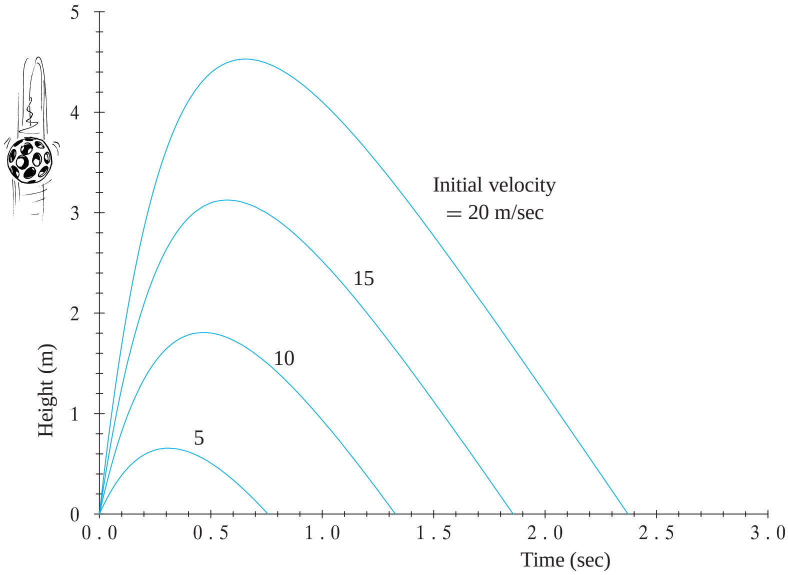


Throw a ball into the air. Does it take longer to go up, or to come down? Check out Example 1.5.5.

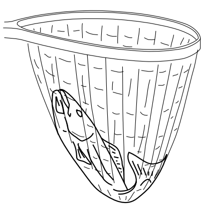
CHAPTER

1

First-Order Differential Equations and Models

How many tons of fish can be harvested each year without killing off the population? When you double the dose of your cold medication, do you fall asleep in your math class? Does it take a ball longer to rise than to fall? In this chapter we model natural processes with differential equations in order to answer these and many other questions.

1.1 A Modeling Adventure



Differential equations provide powerful tools for explaining the behavior of dynamically changing processes. We will use them to answer questions about processes that are hard to answer in any other way.

Take a look, for example, at the fish population in one of the Great Lakes. What harvesting rates maintain both the population and the fishing industry at acceptable levels? We will use differential equations to find out how the population changes over time given birth, death, and harvesting rates.

The clue that a differential equation may describe what is going on lies in the words "birth, death, and harvesting rates." The key word here is "rates." Rates are derivatives with respect to time, but what quantity is to be differentiated in this case? Let's measure the population of living fish at time t by the total tonnage $y(t)$, and the time in years. Then the net rate of change of the fish population in tons of fish per year is $dy(t)/dt$, written as $y'(t)$ or simply y'. At any time t, we have

$$y'(t) = \text{Birth rate} - \text{Death rate} - \text{Harvest rate} \tag{1}$$

where we measure all the rates in tons per year. We suppose that fish immigration and emigration rates from rivers that meet the lake cancel each other out, so they don't need to appear in (1). Close observation of many species over many years suggests that birth and death rates are each roughly proportional to the size of the population:

$$\begin{aligned} \text{Birth rate at time } t&: \quad by(t) \\ \text{Death rate at time } t&: \quad (m + cy(t))y(t) \end{aligned}$$

where b, m, and c are nonnegative proportionality constants. The extra twist here is that the natural mortality coefficient m is augmented by the term $cy(t)$ which accounts for overcrowding. As a population increases in a fixed habitat, the death rate often increases much faster than can be accounted for by a single constant coefficient m. The overcrowding term is needed to model this accelerated mortality factor.

Now let's pull all of these bits and pieces together and create a model.

Making the Mathematical Model

Denoting the harvest rate by H and using the law given by (1), we have a differential equation for $y(t)$:

☞ If H is a positive constant then this model is *constant rate harvesting*.

$$y' = by - (m + cy)y - H$$

or

$$y' = ay - cy^2 - H \tag{2}$$

where $a = b - m$ is assumed to be positive. An equation like (2) that involves a to-be-determined function of a single variable and its derivatives is called an *ordinary differential equation* (ODE, for short).

Referring to our fishing model, we note that observation of an actual fish population gives us a fairly good idea of the birth and death rates (so we suppose that a and c are known), and the harvest rate H is under our control. That leaves the tonnage $y(t)$ to be determined from ODE (2). A function $y(t)$ for which

$$y'(t) = ay(t) - c(y(t))^2 - H$$

for all t in an interval is called a *solution* of ODE (2). The value y_0 of $y(t)$ at some time t_0 can be estimated, and must surely be a critical factor in predicting later values of $y(t)$. The condition $y(t_0) = y_0$ is called an *initial condition*.

☞ We often say that $y(t)$ *satisfies* an ODE when we mean that $y(t)$ is a solution of that ODE.

Measuring time forward from the time t_0, we have created a problem whose solution $y(t)$ is the predicted tonnage of fish at future times:

Mathematical Model for the Fish Population over Time

Given the constants a and c, the harvesting rate H, and the values t_0 and y_0, find a function $y(t)$ for which

$$y' = ay - cy^2 - H, \qquad y(t_0) = y_0 \tag{3}$$

on some t-interval containing t_0.

The ODE and the initial condition in (3) form an *initial value problem* (IVP) for $y(t)$. We will see in Chapter 2 that the general IVP (3) has a unique solution on some t-interval if the harvesting rate H is a constant, or if H is a continuous function of time. It is nice to know that we are dealing with a problem that has exactly one solution, even though we don't yet know how to construct that solution. It is like knowing in advance that the pieces of a jigsaw puzzle will indeed fit together.

So how do we describe the solution $y(t)$ of IVP (3)? Do we use words, graphs, or formulas? We will use all three.

A Solution Formula for IVP (3): No Overcrowding

We have put together a general model IVP for the fish tonnage. To describe the solution, it might be a good idea not to tackle the full-blown initial value problem, but to look at particular cases first.

Suppose that there is no overcrowding (so $c = 0$). Start the clock when the value y_0 is known. This gives us the following IVP: Find $y(t)$ so that

$$y' = ay - H, \qquad y(0) = y_0, \quad t \geq 0 \tag{4}$$

We assume that a, H, and y_0 are nonnegative constants. Here's a way to find a solution formula for IVP (4).

Suppose that $y(t)$ is a solution of IVP (4), that is,

$$y'(t) = ay(t) - H, \qquad y(0) = y_0 \tag{5}$$

Moving all the terms in the ODE of (5) to the left-hand side and multiplying through by e^{-at}, we have that

☞ We explain this approach in Section 1.3.

$$e^{-at}(y' - ay + H) = 0 \tag{6}$$

Since $(e^{-at})' = -ae^{-at}$ and $(e^{-at}y(t))' = e^{-at}y'(t) - ae^{-at}y(t)$, ODE (6) becomes

$$\left(e^{-at}y - \frac{H}{a}e^{-at}\right)' = 0$$

But from calculus we know that the only functions with zero derivatives everywhere are the constant functions. So for some constant C we have

$$e^{-at}y(t) - \frac{H}{a}e^{-at} = C \tag{7}$$

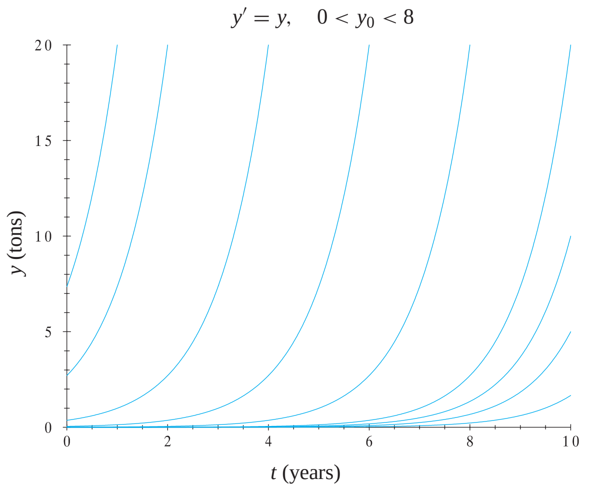


FIGURE 1.1.1 Exponential growth (no harvesting): IVP (4) with $a = 1$, $H = 0$.

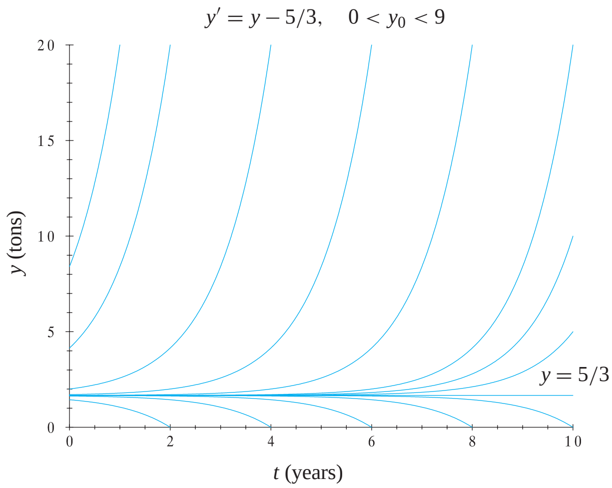


FIGURE 1.1.2 Exponential growth and decline with harvesting: IVP (4) with $a = 1$, $H = 5/3$.

Setting $t = 0$ in formula (7), we can solve for C. We get

$$y_0 - \frac{H}{a} = C \tag{8}$$

since $y(0) = y_0$. So, multiplying each side of formula (7) by e^{at}, using the value for C given in (8), and rearranging terms, we finally see that the solution of IVP (4) has the form

$$y(t) = \frac{H}{a} + \left(y_0 - \frac{H}{a}\right)e^{at}, \quad \text{for } t \geq 0 \tag{9}$$

To complete the construction process you may want to verify that the function $y(t)$ given in (9) actually is a solution of IVP (4).

What does formula (9) tell us about the fish population? First, if initial tonnage y_0 is exactly H/a, then (9) yields $y(t) = H/a$ for all $t \geq 0$. This constant solution $y(t) = H/a$ is called an *equilibrium solution*. Second, note that if y_0 is the slightest bit greater than H/a, then exponential growth sets in. If y_0 is less than H/a, then the fish population becomes extinct since there is a time $t^* > 0$ such that $y(t^*) = 0$.

☞ Keep in mind: a *solution* is a function; a *solution curve* is the graph of a solution.

The graph in the ty-plane of a solution $y(t)$ of an ODE is called a *solution curve*. Figure 1.1.1 shows the exponential growth of the population if there isn't any fishing ($H = 0$). Figure 1.1.2 shows both exponential growth and decline away from equilibrium if there is fishing ($H = 5/3$ tons per year). These two figures can be generated directly by using formula (9) and graphing software.

If $y_0 < H/a$ we soon end up with extinction, but if $y_0 > H/a$, then the fish population grows without bound (which never happens in real life). So we need a better model. Maybe we need to put the overcrowding term back into play.

Overcrowding, No Harvesting

So let's temporarily drop the harvesting term from the ODE and put the overcrowding term back in to obtain the IVP

$$y' = ay - cy^2, \qquad y(0) = y_0, \quad t \geq 0 \tag{10}$$

where a, c, and y_0 are positive constants. Although there is a formula for the solution of IVP (10) (look ahead to Example 1.6.5), the formula isn't particularly easy to derive, so we need another way to describe the solution of IVP (10). There are computer programs called *numerical solvers* that compute very good approximations of the solution to an IVP like (10), even when there is no solution formula. Let's see what we can do with IVP (10) using a numerical solver.

Figure 1.1.3 shows approximate solution curves for IVP (10) with $a = 1$, $c = 1/12$:

$$y' = y - y^2/12, \qquad y(0) = y_0, \quad y_0 = \text{ various positive values}, \quad t \geq 0 \tag{11}$$

We have set the computer solve-time interval at $0 \leq t \leq 10$ to predict future tonnage and the tonnage range at $0 \leq y \leq 20$; negative tonnage makes no sense here.

What does Figure 1.1.3 suggest about the evolving fish tonnage as time advances? First of all, there seem to be two equilibrium levels, $y(t) = 12$ for all $t \geq 0$ and $y(t) = 0$ for all $t \geq 0$. Are these actual solutions of the ODE in (11)? Yes, because the constant functions $y(t) = 12$ and $y(t) = 0$ satisfy the ODE, as can be verified by direct substitution. Intriguingly, the upper equilibrium seems to attract all other nonconstant solution curves in the *population quadrant* $y \geq 0$, $t \geq 0$. Left alone, the fish population tends toward this equilibrium level, no matter what the initial population might be.

Warm up your numerical solver by doing Figure 1.1.3 yourself.

Since we will use numerical solvers often, let's see how they work.

Some Tips on Using a Numerical Solver

A numerical solver plots an approximate value of the solution $y(t)$ at hundreds of different instants of time and then connects these points on the computer screen with line segments. How well this graph approximates the true solution curve depends on the sophistication of the solver. Numerical analysts have done a remarkable job in coming up with reliable solvers; we have a great deal of confidence in ours.

For now we only need to concern ourselves with the basics of how to communicate with the solver. The first thing to do is to write the IVP in the form

$$y' = f(t, y), \qquad y(t_0) = y_0$$

because the numerical solver has to know the function $f(t, y)$ and the *initial point* (t_0, y_0). Since dy/dt is the time rate of change of the solution $y(t)$ of the IVP, the function $f(t, y)$ is often called a *rate function*. Next, the user needs to specify the *solve-time interval* as running from the initial point t_0 to the final point t_1. The IVP is said to be solved *forward* if $t_1 > t_0$, and *backward* if $t_1 < t_0$.

The solver must be told how to display solution curves. We like to select the screen size (i.e., the axis ranges) before telling our solver to find and plot solution curves. There are two reasons for this:

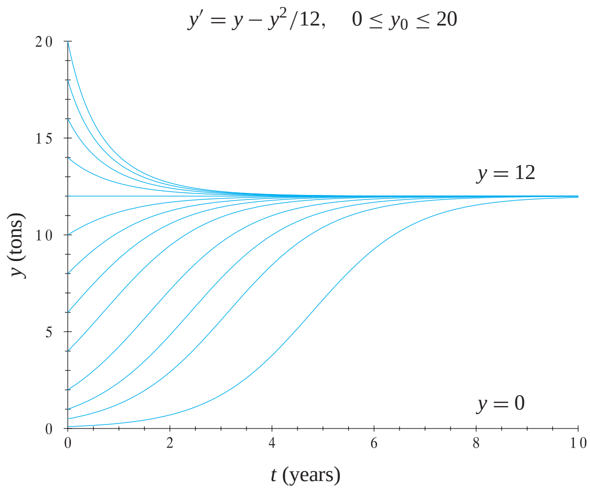


FIGURE 1.1.3 Overcrowding, no harvesting: equilibrium solutions $y = 0, 12$; IVP (11).

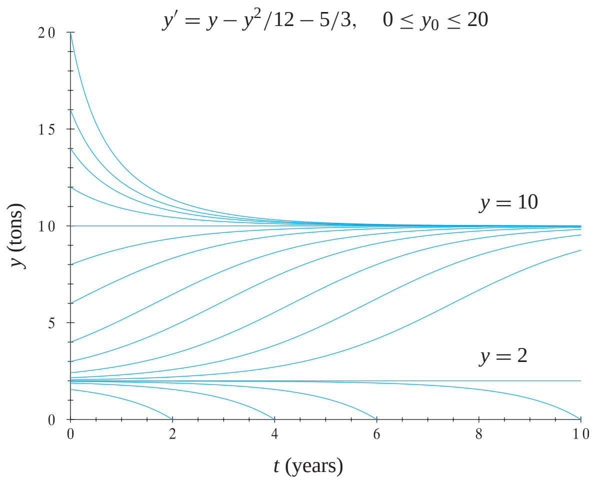


FIGURE 1.1.4 Overcrowding, harvesting: equilibrium solutions $y = 2, 10$; IVP (12).

☞ More tips on using solvers are in the Student Resource Manual.

- Well-designed solvers often shut down automatically when the solution curve gets too far beyond the specified screen area because of a poorly selected solve-time interval. This prevents the computer from working too hard (and perhaps crashing).
- Some solvers have a default setting that automatically scales the screen size to the solution curve over the solve-time interval. If you have a runaway solution curve, you won't see much on the screen.

Choosing the right screen size to bring out the features you wish to examine is often as much of an art as it is a science. Your skill at setting screen sizes will improve with experience.

Now we are ready to return to the fish population model. Let's put the fishing industry back in business and see what happens.

Overcrowding and Harvesting

Let's start out by including light harvesting, say $H = 5/3$ tons per year, so IVP (11) becomes

$$y' = y - \frac{y^2}{12} - \frac{5}{3} = -\frac{1}{12}(y-2)(y-10), \qquad y(0) = y_0 \ge 0 \tag{12}$$

Let's use our solver to plot approximate solution curves to IVP (12) for positive values of y_0 (Figure 1.1.4). There are two equilibrium solutions: $y = 2$ and $y = 10$, all t. The upper equilibrium line still attracts solution curves, but now not all of them. Those starting out below the lower equilibrium line curve downward toward extinction. This model of a low harvesting rate flashes a yellow caution signal: light harvesting doesn't appear to be very harmful, at least if the initial tonnage y_0 is high enough, but even a light harvesting rate could drive a population to extinction if the population level is low

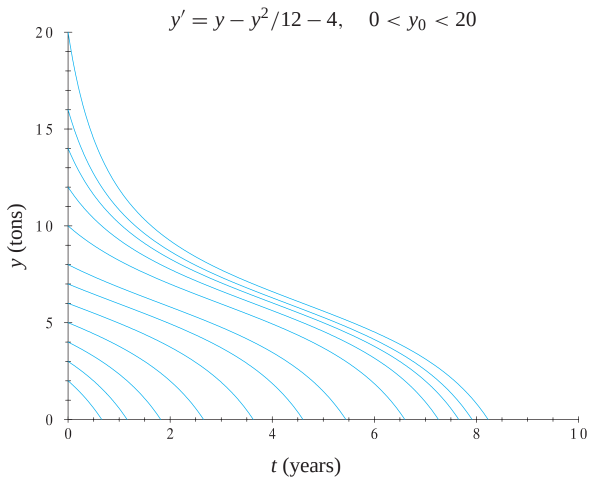


FIGURE 1.1.5 Extinction; IVP (13) for various y_0 values.

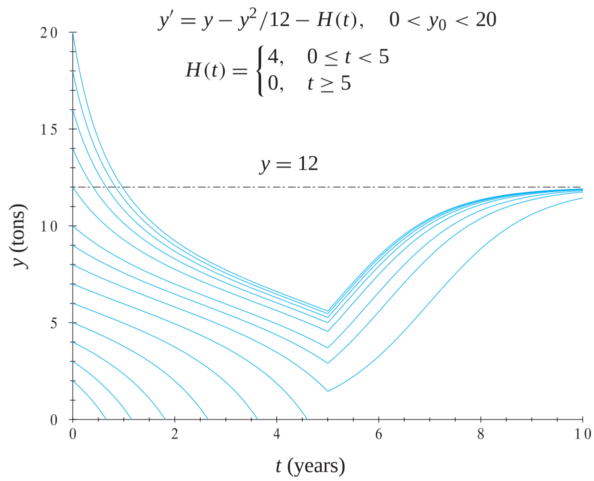


FIGURE 1.1.6 Ban on fishing over a five-year period restores fish population; IVP (14).

to begin with. Still, this is a scenario where both the fish population and the fishing industry do fairly well.

Now let's give the fishermen a free hand and suppose that the harvesting rate is much higher. Let's say the harvesting rate rises to 4 tons per year. We have the heavy harvesting IVP

$$y' = y - \frac{y^2}{12} - 4, \qquad y(0) = y_0, \quad t \ge 0 \tag{13}$$

This time if we search for equilibrium solutions by setting $y' = 0$ and using the quadratic formula to find the roots of $y - y^2/12 - 4$, we find that there are none. In fact, y' is always negative and Figure 1.1.5 shows the resulting catastrophe.

Ban on Fishing

We can't let the fish population die out. Let's see what happens in our model if, after five years of fishing at the rate of 4 tons per year, we ban fishing for five years. Now the harvest rate is given by the function

$$H(t) = \begin{cases} 4, & 0 \le t < 5 \\ 0, & 5 \le t \le 10 \end{cases}$$

and the IVP is

$$y' = y - \frac{y^2}{12} - H(t), \qquad y(0) = y_0, \quad 0 \le t \le 10 \tag{14}$$

Fortunately, it is known that even if the harvesting rate is an on-off function like $H(t)$, an initial value problem such as (14) still has a unique solution $y(t)$ for each value of y_0. We don't have a formula for $y(t)$, but our numerical solver gives us a good idea of just how $y(t)$ behaves.

As you might expect, the fish population is rescued from extinction if y_0 is large. Figure 1.1.6 shows that after five years of heavy harvesting the surviving population heads toward the level of $y = 12$. We have saved the fish, but at the expense of the fishing industry.

Figure 1.1.6 shows a strange feature not seen in any of the other graphs: corners on the solution curves. These appear precisely at $t = 5$ when harvesting suddenly stops. So a discontinuity in the harvesting rate shows up in the graphs as a sudden change in the slope of a solution curve. That is not surprising because the slope of a solution $y(t)$ is the derivative $y'(t)$, and $y'(t)$ in ODE (14) involves the on-off harvesting rate.

Comments

We created a mathematical model using ODEs for changes in population size, a model that includes internal controls (the overcrowding factor) and external controls (the fishing rate). We found formulas for the solutions of the mathematical model in a simple case, used a numerical solver to graph solutions in more complex cases, and interpreted all of these solutions in terms of what happens to the fish population. The model introduced here has its flaws, as all models do. But the modeling process has allowed us to examine the consequences of various assumptions about the rate of change of the fish population.

There are many good solvers that require little or no programming skills. No specific solver is presumed in this text.

PROBLEMS

☞ Underscoring indicates an answer at the end of the book.

☞ Computer icons tell you to use your numerical solver.

1. (*Exponential Growth*). Say that the model IVP for a fish population is given by $y'(t) = ay(t)$, $y(0) = y_0$, where a and y_0 are positive constants (no overcrowding and no harvesting).

(a) Find a solution formula for $y(t)$.

(b) What happens to the population as time advances? Is this a realistic model? Explain.

2. (*Control by Overcrowding and by Harvesting*). The IVP $y' = y - y^2/9 - 8/9$, $y(0) = y_0$, where y_0 is a positive constant, is a special case of IVP (3).

(a) What is the overcrowding coefficient and its units? What is the harvesting rate?

(b) Find the two positive equilibrium levels. [*Hint*: Find the roots of $y - y^2/9 - 8/9$.]

(c) Graph solution curves of the IVP for various values of y_0. Use the axis ranges $0 \le t \le 10$, $0 \le y \le 15$. Interpret what you see in terms of the future of the fish population.

3. (*Restocking*). Restocking the fish population with R tons of fish per year leads to the model ODE $y' = ay - cy^2 + R$, where a and c are positive constants.

(a) Explain each term in the model ODE.

(b) Test the model on the IVP $y' = y - y^2/12 + 7/3$, $y(t_0) = y_0$, for various nonnegative values of t_0 and y_0. Carry solution curves forward and backward in time. Use the screen $0 \le t \le 10$, $0 \le y \le 25$. Interpret what you see.

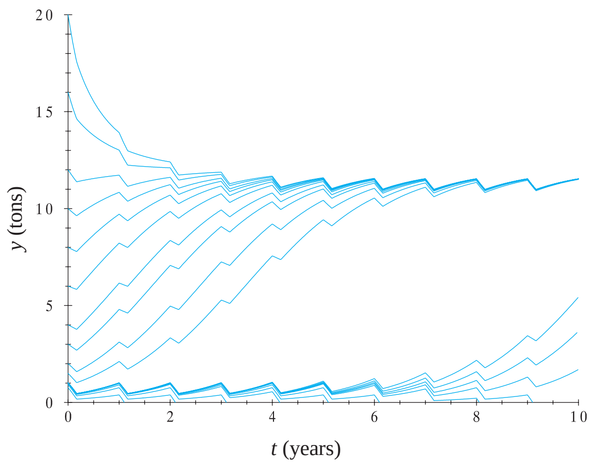


FIGURE 1.1.7 Short harvest season (Problem 7).

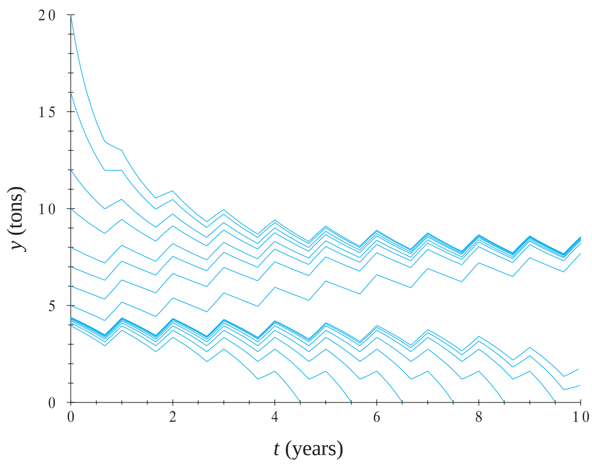
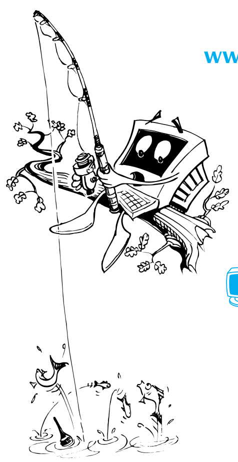


FIGURE 1.1.8 Long harvest season (Problem 7).

4. (*Periodic Harvesting/Restocking*). Look at the IVP $y' = y - y^2 + 0.3\sin(2\pi t)$, $y(t_0) = y_0$.

(a) Discuss the meaning of the ODE in terms of a fish population. Graph solution curves for $t_0 = 0$ and values of y_0 ranging from 0 to 2. Use the ranges $0 \le t \le 10$, $0 \le y \le 2$. Repeat for $t_0 = 1, 2, \ldots, 9$ and $y_0 = 0$. Interpret what you see in terms of the fish population.

(b) Explain why the solution curves starting at (t_0, y_0) and $(t_0 + 1, y_0)$ look alike. In the rectangle $0 \le t \le 10$, $-1 \le y \le 2$, plot the solution curve through the point $t_0 = 0.5$, $y_0 = 0$. Why is this curve meaningless in terms of the fish population?



www **5.** (*Constant Effort Harvesting*). The models in this section have a flaw. At low population levels a fixed high harvesting rate can't be sustained for long because the population dies out. A safer model (for the fish) is $y' = ay - cy^2 - H_0 y$, $y(0) = y_0$, where a, c, H_0, and y_0 are positive constants. In this model the lower the population, the lower the harvesting rate.

(a) Interpret each term in the ODE. Why is this called constant effort harvesting?

(b) For values of H_0 less than a, explain why this model produces solution curves similar to those in Figure 1.1.3, but possibly with a different stable equilibrium population.

6. (*Heavy Harvesting, Light Harvesting*). What happens when a five-year period of heavy harvesting is followed by five years of light harvesting? Combine IVPs (12) and (13) by supposing that $y' = y - y^2/12 - H(t)$, where

$$H(t) = \begin{cases} 4, & 0 \le t < 5 \\ 5/3, & 5 \le t \le 10 \end{cases}$$

Plot solution curves for $0 \le t \le 10$, $0 \le y \le 20$, and interpret what you see. Draw the lines $y = 10$ and $y = 2$ on your plot and explain their significance for the population for $t \ge 5$.

7. (*Seasonal Harvesting*). Say that harvesting is seasonal, "on" for the first few months of each year and "off" for the rest of the year. The ODE is $y' = ay - cy^2 - H(t)$, where $H(t)$ has value H_0 during the on-season, and value 0 during the off-season. The harvesting season is the first two months of each year in Figure 1.1.7 and the first eight months in Figure 1.1.8; in both figures $a = 1$, $c = 1/2$, $H_0 = 4$. Duplicate the graphs in Figures 1.1.7 and 1.1.8. Discuss what you see in terms of population behavior. [*Hint*: Try $H(t) = H_0\,\text{sqw}(t, d, 1)$, where $d = 100(2/12) = 50/3$ for a two-month season and $d = 100(8/12)$ for an eight-month season. See Appendix B.1 for more information about the on-off function sqw.]

If you run into trouble, refer to "Tips" in the Student Resource Manual.