e : A Conceptually Defined Number:

its curious and remarkable properties

Richard W. Beveridge

The Number e

The number e is not so easily defined as its cousin π. While π does possess complex properties, its simplest expression is as the ratio between a circle's diameter and its circumference. On the other hand, e is the value of the base for which the ratio between the slope of the exponential function and the function value itself is one. This is a somewhat more complex proposition.

The value of e lies between 2 and 3, with an approximate value of 2.718281828. The fact that e is a transcendental number means that it is not the root of any polynomial equation with integer coefficients. Euler proved that e was irrational in 1737. The proof that e is transcendental was accomplished by Hermite in 1873. The conceptualization of e really began with the early development of the calculus in the 17th century, but John Napier did some work previous to this that touched on the idea of e as well.

Napier's Logarithms

The Mathematics History website at the University of St. Andrews says that "it is not unreasonable (but perhaps a little misleading) to say that [Napier's logarithmic tables] are to base $\frac{1}{e}$." Napier devised his logarithmic tables so that a given number N was expressed in the form $N = 10^7(1-10^{-7})^L$, where the value of L was known as the Napierian logarithm of N. If we consider one of the presently accepted definitions of e, that of $\lim_{x\to\infty}(1+\frac{1}{x})^x$, then we can see that Napier's calculations can lead to an expression of $(1-\frac{1}{10^7})^{10^7}$ whose value is quite close to $\frac{1}{e}$. The expression $\frac{1}{(1+\frac{1}{10^7})}$ and $(1-\frac{1}{10^7})$ differ by $\frac{1}{10^7(10^7+1)}$.

So, as these two expressions are nearly reciprocals of each other, and the value of 10^7 is large enough to give a fairly good approximation for e, then we can see that Napier's work does explore the outskirts of e's territory.

In finding the effects of compound interest on a principal amount P, we can use the equation $A = P(1+\frac{r}{n})^{nt}$, where r is the interest rate, n is the number of compounding periods per year and t is the amount of time the money is invested. In abstracting this expression, we might consider a period of 1 year and set $r = 100\%$ to represent a doubling of the value of the principal. If we then increase the number of compounding periods to infinity we can examine the effect this has on the value of the investment. The resulting expression will be $(1+\frac{1}{x})^x$. In 1683, Jacob Bernoulli worked to find the limit of $(1+\frac{1}{x})^x$ as x goes to infinity and concluded that it lay between 2 and 3.

<u>The Quadrature of the Hyperbola</u>

Early attempts to find the area under a curve, or what was historically known as quadrature, led to the establishment of series representations for $\ln(1+x)$ and $e^x - 1$. In 1635, the Italian monk Bonaventura Cavalieri discovered a formula for the area under a polynomial curve, or what is known in modern notation as $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$. He conjectured that this was true for all positive integers n. This result was proven several years later by Fermat, Descartes and Roberval. Cavalieri also discovered that the solid obtained by revolving the curve $\frac{1}{x}$ about the x-axis from 1 to ∞ has a finite volume. The philosopher Hobbes said of this, "To understand this for sense, it is not required that a man should be geometrician or logician, but that he should be mad."

It was unfortunate that this formula of Cavalieri and the others would not work for x^{-1}, as division by 0 is undefined. However, Gregoire St. Vincente, a Belgian Jesuit, studied this special case extensively and concluded that the area under the hyperbola was a logarithmic function, although no base was specified. St. Vincente's main conclusion was that, as the upper terminus of integration was extended geometrically, the area under the curve increased arithmetically. From St. Vincente's work the "natural logarithm" could be defined in terms of the integral $\int_1^x \frac{1}{t} dt$.

The series representation for the expression $\frac{1}{1+x}$ can be obtained through long division and was certainly known to mathematicians of the 17th century. Isaac Newton provided an ingenious generalization of the binomial theorem to allow expansion of expressions raised to powers that were negative integers. To do this, he found a pattern in the Pascal Triangle representation of the coefficients that allowed him to infer what the coefficients would be for negative integer exponents. William Brouncker, Nicolas Mercator (not the map maker) and Newton all arrived at infinite series representations for $\ln(1+x)$ through the term-by-term integration of the series for $\frac{1}{1+x}$. Newton then chose an arbitrary "end" for the series and inverted it to determine that the inverse of the function determined by $\int_1^x \frac{1}{1+t} dt$ could be expressed as the infinite series $x = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots$, which is clearly the series representation for $e^z - 1$.

Huygens, Leibniz and the Catenary

Around 1647, the Italian mathematician Evangelista Torricelli began studying the curve that, at the time was known as the logarithmica. Today this is known as the exponential curve. Using classical Greek geometric methods, Torricelli demonstrated that the "subangent," which is the ratio of the y-coordinate to the slope of the curve, is a constant. He also demonstrated that the area under the curve between two x-coordinates is the difference between the y-coordinates multiplied by the sub-tangent.

During the early 1660's, Christiaan Huygens also worked with the logarithmica curve. He presented his findings and applications to the Paris academy in 1669, but did not publish them until 1690. Torricelli's work was not published until 1900. Consequently, the first published work on the exponential curve, or logarithmica, was that of the Scotsman James Gregory in 1667. Gregory had spent the previous five years in Italy with Torricelli's pupil Stefano degli Angeli, so his work was possibly based on that of Torricelli's.

By 1690, mathematicians had taken great interest in what is now known as the catenary curve. A hundred years before, Galileo had proposed that the parabolic motion of a falling object represented the same curve as that of hanging chain. In 1669, Joachim Jungius had shown this to be false.• It was in 1690 that Jacob Bernoulli challenged the mathematical community to solve the problem of

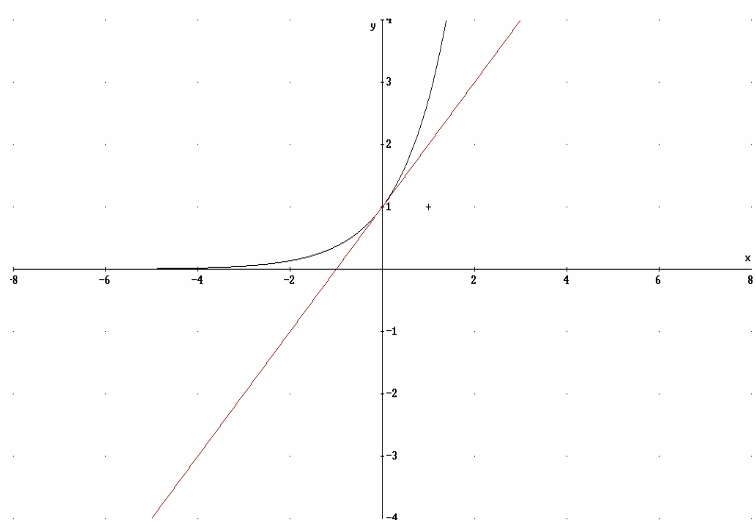
• In *e:The Story of a Number*, Eli Maor states that Huygens disproved this theory in 1642, but quotes no reference, and no other source mentions this possibility.

the hanging chain. About a year later, three independent solutions were published in Leibniz's academic journal *Acta eruditorum*. One was by Leibniz himself, one by Huygens, and one by Johann Bernoulli, Jacob's younger brother. Leibniz's solution recognized that the catenary was related to the exponential (logarithmica) curve and to the quadrature of the hyperbola as well.

The equation for the catenary in modern notation is $\dfrac{e^x + e^{-x}}{2}$.

_e_ and the Calculus

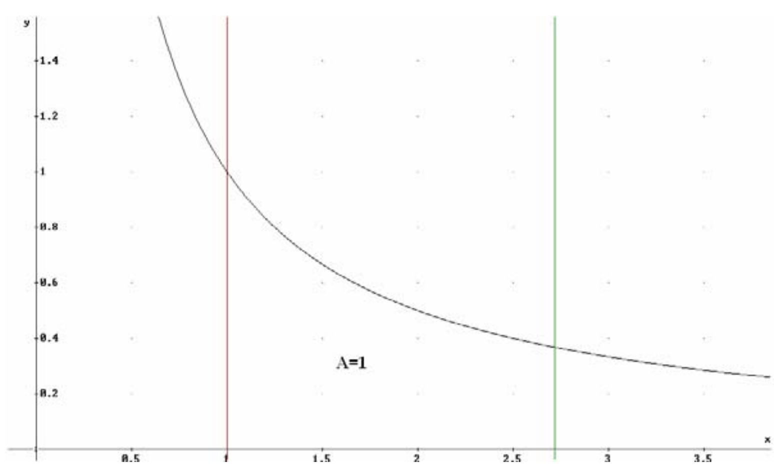
Torricelli's work had shown that the slope of the exponential function is proportional to the functional value itself. Naturally, the question arose as to which value of the base would cause the slope and the function to have the *same* value. In other words, for what value of b would $\lim_{h \to 0} \dfrac{b^h - 1}{h} = 1$?



It can be shown that the value of b that gives this result is the same as the value of the expression $\lim_{x \to \infty}(1+\frac{1}{x})^x$.

Given $\lim_{h \to 0}\frac{b^h - 1}{h} = 1$, as $h \to 0$, then for the ratio to tend to 1, $b^h - 1$ should approach h. Thus, $b^h - 1 \to h$, so $b^h \to h+1$, and $b \to (1+h)^{\frac{1}{h}}$, thus $b = \lim_{h \to 0}(1+h)^{\frac{1}{h}}$. Using the substitution $h = \frac{1}{x}$, then as $h \to 0$, $\frac{1}{x} \to 0$, and $x \to \infty$, so $b = \lim_{x \to \infty}(1+\frac{1}{x})^x$.

In the problem of the quadrature of the hyperbola, the quantity $\lim_{x \to \infty}(1+\frac{1}{x})^x$ is also the quantity that gives a value of one to the expression $\int_1^x \frac{1}{t}dt$.



If we consider $\int_1^{\lim_{x\to\infty}(1+\frac{1}{x})^x} \frac{1}{t}\,dt$, then to arrive at a value for this expression, using the modern understanding of ln, we can calculate $\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x) - \ln 1$, which is clearly 1. Without using modern methods to calculate $\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x)$, there is a somewhat more roundabout way to show that $\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x) = 1$, however it requires that we know the derivative of $\ln x$.

As is generally the case in mathematics, it is typically easier to understand a concept if we examine it in a manner different from the way in which it was initially considered. In considering the function $f(x) = \ln x$, it is much simpler to differentiate $\ln x$ to arrive at $\frac{1}{x}$ than it is to integrate, or find the quadrature of $\frac{1}{x}$. It is not very complex to show that if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. There are two ways to do this, one involves the limit definition of the derivative and the other involves implicit differentiation.

Using the limit definition…

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{1}{h} * \ln\left(1+\frac{h}{x}\right) =$$

$$\lim_{h\to 0} \frac{1}{x} * \frac{x}{h} * \ln\left(1+\frac{h}{x}\right) = \frac{1}{x} \lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \frac{1}{x} * \ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right).$$

If we let $n = \frac{x}{h}$, then as $h \to 0$, $n \to \infty$.

$$f'(x) = \frac{1}{x} * \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = \frac{1}{x} * \ln e = \frac{1}{x} * 1 = \frac{1}{x}$$

If we use implicit differentiation…

$y = \ln x$, so $e^y = x$. Differentiating with respect to x…

$e^y = x$, then $y' * e^y = 1$. So $y' = e^{-y} = e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$.

$y' = \frac{1}{x}$.

Now that we know the derivative of $\ln x$, we can show another method of calculating the numerical value of $\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x)$. If $f(x) = \ln x$, then

$f'(x) = \frac{1}{x}$, and $f'(1) = 1$. So,

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x} * \ln(1+x) = \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}$$

$$= \ln\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\} = f'(1)$$, which we already know is 1.

Therefore, $\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x) = 1$, so $\int_1^{\lim_{x\to\infty}(1+\frac{1}{x})^x}\frac{1}{t}\,dt = 1$.

The Relationship between e and i

During the first decade of the 18th century, an English mathematician named Roger Cotes worked at editing Newton's *Principia Mathematica*. During this time, he also researched ideas that were related to Newton's work and other ideas as well. In the only paper published during Cotes' lifetime, *Logometria* (1712), we find the series representations for $\ln(1+x)$ and $\ln(1-x)$, continued fractions for e and $\frac{1}{e}$, as well as calculations of e and $\frac{1}{e}$ to twelve decimal places.

In a paper published posthumously in 1722, *Harmonia Mensurarum*, Cotes approached two important results that would be fully realized later in the 18th century. Applying trigonometric methods to the unit circle, Cotes developed an early geometric representation for the roots of unity. In a related vein of inquiry, he used substitution involving the logarithmic and arctangent functions to arrive at two equivalent expressions for the same integral. The results of Cotes' substitution can be expressed in modern notation as $\ln(\cos\theta + i\sin\theta) = i\theta$. In the mid-1800's Euler would achieve the definitive result (as he often did).

By the time of Euler, the series representations for e^x, $\sin x$, and $\cos x$ were well known. Euler used the series expansions on each side of the equation $e^{i\theta} = \cos\theta + i\sin\theta$ to show that it was indeed true. If we assume that the series for e^x will hold true for imaginary values of x as well as real values, then we can represent $e^{i\theta}$ in the following manner,

$$1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \ldots$$

We can divide this series into a real component and an imaginary component to see that it is a combination of the sine series and the cosine series…

$$=\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)=\cos\theta+i\sin\theta.$$

Using calculus, we may show the same equality. If we set $z=\cos\theta+i\sin\theta$ and differentiate, then

$$\frac{dz}{d\theta}=-\sin\theta+i\cos\theta=i(\cos\theta+i\sin\theta)=iz$$

So, now we have $\dfrac{dz}{d\theta}=iz$, which implies that $\dfrac{dz}{z}=id\theta$.

Integrating on both sides, we have $\int\frac{dz}{z}=i\int d\theta$, consequently $\ln z=i\theta+C$. If we use the boundary condition that when $\theta=0$, $z=1$, then we can conclude that $C=0$. So, our equation now becomes $\ln z=i\theta$, or

$$e^{i\theta}=z=\cos\theta+i\sin\theta.$$

From this relationship comes Euler's equation relating five of the most important mathematical constants…

$$e^{i\pi}+1=0.$$

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